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Classical and Quantum Noise in Nonlinear Optical Systems

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In quantum optics noise plays an important role, since many of the nonlinear optical systems are quite sensitive to the subtle influences of weak random perturbations, being either classical of quantum mechanical in nature. We discuss the origin of quantum noise emerging from the reversible or the irreversible part of the dynamics and compare it with the properties of purely classical fluctuations. These general features are illustrated by a number of physical examples, such as the laser with loss or gain noise, nonlinear optical devices, and the phenomenon of quantum jumps. These processes have been chosen mainly to illustrate the different aspects of noise, but also because, to a large extent, they can be described in analytical terms.

KEY WORDS: Fokker–Planck equation; Glauber P-representation; positive P-representation; photon statistics; sub-Poisson statistics; squeezing.

1. INTRODUCTION

In descriptions of classical macroscopic systems, the physical properties can essentially be characterized by deterministic dynamic equations, and fluctuations or noise play only a minor role. Thus, only when one looks very closely are the inevitable and minute irregularities observed. In the field of quantum optics, this seems to be somewhat different, since from the early days of laser physics, the question of noise has played an essential role. Due to the enormous precision that can be achieved in optical experiments, the effects of randomness are often remarkable and can seldom be neglected.

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In the case of the laser, it is of fundamental interest to determine the mechanisms that limit the possible reduction of the linewidth, and thereby determine the limits of accuracy of an optical experiment.

There exists a number of nonlinear optical devices which exhibit instabilities and bifurcations similar to phase transitions in equilibrium thermodynamics.[1,2] In such cases noise is important, since close to a transition point, where two states compete for and finally exchange the stability, the restoring forces become weak and can no longer supress the effects of fluctuations. This is a general feature not only in quantum optics, but in many other areas, such as chemical reaction dynamics and hydrodynamics. In the neighborhood of an instability, fluctuations tend to grow to macroscopic order, where they may even dominate the dynamic behavior. In those cases, it is most obvious that a statistical description must be applied and noise taken into account explicitly.

Fluctuations can arise from various sources. For example: in thermodynamic equilibrium systems, thermal fluctuations, which are proportional to the temperature and scale inversely proportional to the system volume, can also play a role in quantum optics. To be effective, however, the photon energies must be roughly of the same order as the thermal energy kT. This is the case, for instance, for microwave transitions in Rydberg atoms even at very low temperatures. A large variety of random influences can be tracked back to more or less technical reasons, such as the mechanical stability and the vibrations of optical resonators, fluctuations in the plasma of a laser discharge, hydrodynamic fluctuations in a dye-jet, or the intensity fluctuations of the driving laser field itself. All these random perturbations are classical in nature and lead to a semiclassical description. While most of the previously mentioned noise sources might be eliminated at least in principle, the quantum nature of the underlying microscopic processes introduces noise that is inevitable and inherent to all quantum mechanical problems. While most macroscopic systems seldom are sensitive enough to feel such subtle perturbations, in quantum optics one finds an increasing number of examples where quantum noise plays the essential role.

Quantum fluctuations can appear in various forms, depending on whether they are generated by the reversible or the irreversible part of the dynamics. Interacting with a reservoir, a quantum system can either be subject to dissipation, when the reservoir is kept in thermal equilibrium, or it may be driven far from equilibrium, when the reservoir, loosely speaking, is prepared at negative temperatures, representing a pumping process. In both cases, the reservoir coupling introduces noise into the system, which is qualitatively identical with the effects of classical fluctuations, and which even persists for vanishing temperatures in the second case. Also, the reversible Hamiltonian dynamics themselves can introduce noise, in general through nonlinear interactions. But this sort of randomness has rather peculiar properties, which have only little if anything in common with a classical stochastic process. On the one hand, this makes these processes rather interesting, because we can expect some new and interesting properties that are unknown in classical physics. On the other hand, the mathematical formulation of these effects can become rather difficult and may require the development of new methods.

Among the basic quantum optical processes, there is a growing number of examples that require a statistical description due to the presence of either classical or quantum noise: the laser with intrinsic spontaneous emission noise; the dye-laser with external pump-field noise; optical bistability with a noisy driving field; parametric processes, sub- and second harmonic generation exhibiting quantum noise; resonance fluorescence as a source of light with typical nonclassical properties; and quantum jumps in three-level atoms are triggered purely by quantum noise, but still are visible to the naked eye.

The paper is organized as follows: In the next section we briefly summarize the relevant physical quantities or observables that are useful for characterizing the statistical behavior of optical systems. Section 3 reviews the different mathematical approaches that are available to actually calculate the properties mentioned above for a given quantum optical problem. Thereby we will illustrate the different properties of quantum fluctuations. The last section summarizes a number of specific examples and their solutions, chosen to illustrate the different aspects of classical or quantum, additive or multiplicative noise in optical systems.

2. PHYSICAL OBSERVABLES

Generally speaking, quantum optics is the science of the interaction of light and matter. In a more limited sense, this term is restricted to the interaction of matter with the quantized electromagnetic field only. In both cases, namely in the semiclassical and the quantum formulations, physical observables are described by ensemble averages that are taken over the externally imposed classical or the intrinsic quantum fluctuations. In this sense, the ensemble averages that appear in the following either represent an integral over a positive classical probability density or the trace involving the density operator in the quantum case. The averages can even stand for both, when classical as well as quantum noise are present simultaneously.

Experimentally, we either measure the properties of the field, such as amplitudes, intensities, and their correlations, or we observe directly the atomic properties, such as the population distribution over the energy levels. The field $E(x, t)$ in many examples can be expanded in a sum over the eigenmodes of a fictitious cavity:

$$E(x, t) = \sum_{k,j} \left(\frac{2\pi h\omega}{Vc^2}\right)^{1/2} (e_{kj} b_{kj}^{\dagger} e^{i\omega_k t} + e_{kj} b_{kj} e^{-i\omega_k t}) \tag{2.1}$$

where e_{kj} is the polarization vector and ω_k is the frequency of the mode with wave vector k, V is the quantization volume and c the speed of light. For many phenomena of interest, we have to consider only a small number of modes that interact dynamically. In quantized form, $E(x, t)$ becomes an operator-valued field, while the amplitudes $b_{kj}^{\dagger}$ and b_{kj} turn into the corresponding boson operators.

A hetero- or homodyne experiment allows one to measure the field amplitude directly, by beating it with a reference field, the local oscillator, on the cathode of a photon multiplier. Thereby it is possible to obtain phase-sensitive information about the field. As a function of the relative phase we may observe

$$\mathrm{Re}(E) = \frac{1}{2} \langle b^{\dagger}(t) + b(t) \rangle, \qquad \mathrm{Im}(E) = \frac{1}{2i} \langle b^{\dagger}(t) - b(t) \rangle \tag{2.2}$$

or any linear combination. The angle brackets denote the classical or quantum average. The intensity of the field is given by

$$I(t) = \langle b^{\dagger}(t)\, b(t) \rangle \tag{2.3}$$

These averages evolve in time and represent the transient behavior of a system that generally relaxes, from an initially prepared arbitrary state, toward its stationary state. This state may be time independent or may oscillate periodically in time, when an external field breaks the time-reversal symmetry. A quantitative measure for the randomness of these fields is provided by the appropriate variances. For example, the random excursions of the field amplitude are characterized by

$$\begin{aligned} \langle \Delta\, \mathrm{Re}(E)^2 \rangle &= \frac{1}{4} [\langle (b^{\dagger} + b)^2 \rangle - \langle b^{\dagger} + b \rangle^2] \\ \langle \Delta\, \mathrm{Im}(E)^2 \rangle &= -\frac{1}{4} [\langle (b^{\dagger} - b)^2 \rangle - \langle b^{\dagger} - b \rangle^2] \end{aligned} \tag{2.4}$$

and the intensity fluctuations are measured in terms of

$$\langle \Delta I^2 \rangle = \langle b^{\dagger} b^{\dagger} b b \rangle - \langle b^{\dagger} b \rangle^2 \tag{2.5}$$

For a classical stochastic process, the sequence of $b^\dagger$ and b is irrelevant in Eq. (2.5), but in the corresponding quantum case normal ordering of the boson operators becomes essential, in order to be sensitive to the two photon properties only.

A stationary ensemble is independent of the arbitrariness of the initial preparation, and therefore single time averages must be contant and in general are identical with the asymptotic values of the transient moments above. The intrinsic dynamical properties of the stochastic process are characterized by stationary correlation functions. The phase-sensitive amplitude correlation function, for example, contains information about the spectral properties of the field and its linewidth:

$$G_1(t) = \langle b^\dagger(t)\, b(t=0) \rangle \tag{2.6}$$

while the two-photon correlation function describes temporal excursions of the field intensity or photon number correlations:

$$G_2(t) = \langle b^\dagger(t=0)\, b^\dagger(t)\, b(t)\, b(t=0) \rangle \tag{2.7}$$

Intuitively speaking, G_2 is proportional to the joint probability of observing a first photon at $t=0$ and a second one a time t later. It is not an exclusive probability, in the sense that the second photon is precisely the next one emitted, since there can actually be an arbitrary number of photons in between.

A statistical quantity that provides an exclusive measure can be found in the photon counting distribution. That is, the probability that in a given time interval T a photon detector with quantum efficiency η will detect n and only n events[3,4] is given by

$$W(n, T) = \operatorname{tr} \rho\, \hat{T}\left(\frac{1}{n!}\left[\eta \int_0^T b^\dagger b\, dt\right]^n \exp\left[-\eta \int_0^T b^\dagger b\, dt\right]\right) \tag{2.8}$$

where $\hat{T}$ is an approriate ordering operator and ρ the density operator of the stationary state. The average recorded photon number is given by

$$\langle n \rangle = \sum_{n=0}^{\infty} n W(n, T) \tag{2.9}$$

while the photon number fluctuations may be characterized by Mandel's Q

parameter,[5,6] which measures the deviation of the actula variance from that of the corresponding Poissonian distribution:

$$Q = \frac{\sum_{n=0}^{\infty} (n - \langle n \rangle)^2 \, W(n, T)}{\langle n \rangle} - 1 \tag{2.10}$$

$Q = 0$ indicates that the variance of a given process is numerically indentical to that of a Poisson process, without necessarily being one. $Q > 0$ indicates super-Poissonian statistics and $Q < 0$ sub-Poissonian.

The definition of $W(n, T)$ above is explicitly given in quantum mechanical notation. There does exist, however, a classical counterpart, which we obtain by replacing the operators by classical mode amplitues and the average by an integral over the stationary probability density. It is not a contradiction in itself to talk about a "classical photon distribution," since by the term photons we merely mean the discrete events recorded by a photon multiplier and not the quantized states of the field. In this sense, the term classical means that we neglect the quantum fluctuations of the field, but nevertheless, the counting events are taken to be discrete.

Starting from the counting probability $W(n, T)$, one can derive further probability measures that are quite useful for the comparison of theoretical and experimental results, such as:

1. The probability of observing no photons over a time interval $0 < t < T$ after an observation of a photon at $t = 0$ is

$$P_1(T)\, dT = -\frac{dW(n=0, T)}{dT}\, dT \tag{2.11}$$

2. The probability of observing a dark period of precisely the length T, sandwiched between two photon events at $t = 0$ and $t = T$, is

$$P_2(T)\, dT = -\frac{1}{P_1(T)} \frac{d^2 W(n=0, T)}{dT^2}\, dT \tag{2.12}$$

From the definition and the intuitive understanding of the counting probability $W(n, T)$, it is quite obvious that P_1 and P_2 represent positive probability densities with the appropriate physical meaning.

3. CLASSICAL AND QUANTUM FLUCTUATIONS

In this section we will briefly discuss the similarities and the differences in the physical properties of fluctuations that are either of classical or of quantum mechanical origin. We also discuss the different mathematical methods that are available for describing those dynamic processes in detail.

3.1. Semiclassical models

An amazingly large number of problems in quantum optics do not require the quantization of the electromagnetic field, but can be described by semiclassical theory with satisfactory accuracy. Most external sources of randomness and all the technical fluctuations in the mechanical setup of an experiment fall into this category. Fluctuations can also arise in the dynamic equations of the atoms, which then are transfered to the field, in a way that the field can experience the effects of quantum noise without being quantized itself. Semiclassical models are typically derived in the following way.

The dynamics of matter is described by a master equation for the atoms under the influence of a classical field $E(x, t)$:

$$\frac{d\rho_A}{dt} = -\frac{i}{h}[H, \rho_A] + \left(\frac{d\rho_A}{dt}\right)_{\text{irr}} \tag{3.1}$$

where the Hamiltonian depends on the field amplitude $E(x, t)$ and ρ_A is the statistical operator for the atoms. The electromagnetic field satisfies Maxwell's equations, with the average atomic polarization as the driving force:

$$\frac{\partial E(x, t)}{\partial x} + \frac{1}{c}\frac{\partial E(x, t)}{\partial t} = \frac{4\pi}{c} \operatorname{tr} \rho_A P \tag{3.2}$$

where P is the operator of the resonant atomic polarization and $E(x, t)$ is the slowly varying complex field amplitude. The statistical operator ρ_A is a functional of the electromagnetic field $\rho_A = \rho_A(E(x, t'))$ containing E at all earlier times t'. In the adiabatic approximation, i.e., when the atomic dynamics evolves much faster than the transients of the field itself, the retardation in ρ_A may be neglected and the functional on the right-hand side of the field equation turns into a simple instantaneous function of $E(t)$ directly:

$$\frac{4\pi}{c} \operatorname{tr} \rho_A P = F(E(x, t)) \tag{3.3}$$

Under this assumption we obtain a closed and in general nonlinear equation for the complex field amplitude alone, which typically is of the form

$$\frac{dE(t)}{dt} = \left(-\gamma + \frac{\Gamma}{1 + |E(t)|^2}\right) E(t) + \cdots \tag{3.4}$$

The inclusion of classical noise then will change this equation into a Langevin equation with various random force $\xi_j(t)$. When we expand the field about zero we find

$$\frac{dE(t)}{dt}=[d-|E(t)|^2]\,E(t)+\cdots+\xi_1(t)+\xi_2(t)\,E(t)+\cdots \tag{3.5}$$

where ξ_1 is a sum over the contibutions of all additive random influences, such as thermal or spontaneous emission noise. At optical frequencies thermal noise is negligible and the main source of additive noise comes from the vacuum fluctuations. ξ_2 represents a typical multiplicative source of noise, such as the fluctuations in loss or gain. A peculiar source of multiplicative fluctuations is the quantum noise that arises in nonlinear dynamic systems. If the correlation time of $\xi_j(t)$ is too short to be of any physical relevance or just experimentally inaccessible, we may approximate this problem by a stationary, Gaussian Markov process and assume

$$\langle \xi_i(t)\,\xi_j(t')\rangle = Q_i\delta_{ij}\delta(t-t') \tag{3.6}$$

Statistically equivalent to this description is the formalism of the Fokker–Planck equation for the probability density $P(E, t)$, that is, the probability of observing the field in the interval between E and $E+dE$ at a certain time t:

$$\begin{aligned}\frac{\partial P(E,t)}{\partial t}=&-\frac{\partial}{\partial E}\{(d-|E|^2)\,E+\cdots\}\,P(E,t)\\&-\frac{\partial}{\partial E^*}\{(d-|E|^2)\,E^*+\cdots\}\,P(E,t)\\&+Q_1\frac{\partial^2}{\partial E\,\partial E^*}P(E,t)+Q_2\frac{\partial^2}{\partial E\,\partial E^*}|E|^2\,P(E,t)+\cdots\end{aligned} \tag{3.7}$$

Here we have chosen the Stratonovich interpretation of the stochastic process. The stationary solution $P_0(E)$ of this equation determines the long-time behavior, independent of the chosen initial condition, and allows one to calculate all relevant stationary moments:

$$\lim_{t\to\infty}\langle E^n(t)\rangle=\int_{-\infty}^{+\infty}E^nP_0(E)\,dE \tag{3.8}$$

and the field variance such as

$$\langle \Delta E^2\rangle=\langle E^2\rangle-\langle E\rangle^2=\int_{-\infty}^{+\infty}(E-\langle E\rangle)^2\,P_0(E)\,dE>0 \tag{3.9}$$

The variances for all classical processes with a positive probability density $P_0(E)$ are necessarily positive quantities. This may become different when we investigate processes where the quantum noise dominates over the classical fluctuations.

The time-dependent solution of the Fokker–Planck equation is found systematically in the form of an eigenfunction expansion:

$$P(E, t) = \sum_{n=0}^{\infty} c_n P_n(E) \exp(-\lambda_n t) \tag{3.10}$$

where P_n and λ_n are the eigenfunctions and eigenvalues, respectively, subject to appropriate boundary or integrability conditions. In general the spectrum may be partially discrete and partially continuous. In such a case Eq. (3.10) contains a sum over the discrete and an integral over the continuous branch at the spectrum. c_n is determined by the initial condition. For complex fields $E = re^{i\phi}$ it is useful to separate the eigenfunctions into modulus and phase, especially when the problem considered is phase invariant:

$$P_{n,m}(E) = P_n^m(r)\, e^{im\phi}, \qquad \lambda = \lambda_n^m \tag{3.11}$$

With these definitions, the stationary field correlation function, for instance, assumes the following form:

$$G_1(t) = \iint rr' \{\exp[i(\phi - \phi')(m-1)]\}\, P_n^m(r)\, P_n^m(r') \times \exp(-\lambda_n^m t)\, dr\, dr'\, d\phi\, d\phi' \tag{3.12}$$

which apart from a rapid initial transient is mainly proportional to the exponential decay of the slowest "phase-sensitive" eigenvalue:

$$G_1(t) \to \exp(-\lambda_0^1 t) \tag{3.13}$$

while the intensity correlation function $G_2(t)$ in the same limit is essentially determined by

$$G_2(t) \to \exp(-\lambda_1^0 t) \tag{3.14}$$

provided that the lower part of the spectrum is discrete. From the obvious fact that the probability $P(E, t)$ is a positive-definite function, one easily derives the following general inequalities:

(i) The inequality

$$G_2(t) \geqslant 0 \tag{3.15}$$

tells us that antibunching is impossible in a classical ensemble.

(ii) The photon counting probability,

$$W(n, T) = \frac{1}{n!} \int_0^{\infty} (\eta I T)^n\, e^{-\eta I T} P_0(I) \cdot 2\pi I\, dI \tag{3.16}$$

is always wider than the Poissonian distribution, since P_0 can, at the most, represent a deterministic process, by assuming the form of a delta function, i.e., $\delta(I-I_0)$ and additional noise can only widen the distribution. Therefore Mandel's Q-parameter for a classical process is always limited to positive values:

$$Q \geqslant 0 \tag{3.17}$$

Classical photon statistics is bound to results in super-Poissonian distributions and therefore sub-Poissonian densities can only exist in the quantum limit, where we must abandon the requirement of positive definiteness.

(iii) The variances of the quadrature components of the field X^+ and X^- have a general lower bound

$$\begin{aligned}\langle(\Delta X^+)^2\rangle &= \left\langle\left[\frac{1}{2}(E^\dagger + E)\right]^2 + \frac{1}{4}\right\rangle \geqslant \frac{1}{4}\\ \langle(\Delta X^-)^2\rangle &= \left\langle\left[\frac{1}{2i}(E^\dagger - E)\right]^2 + \frac{1}{4}\right\rangle \geqslant \frac{1}{4}\end{aligned} \tag{3.18}$$

which means that in a phase-sensitive experiment, the variance are necessarily larger than 1/4 and a classical system will never exhibit squeezing as is possible in the quantum domain. The last definition above will become more transparent when formulated directly in quantum mechanical terms.

3.2. Quantum Mechanical Models

For the present purposes we will assume that the atomic coherence and the excited state populations decay on a short time scale, much shorter than the time scales on which the field evolves. This enables us to eliminate the atomic dynamics adiabatically—as in the semiclassical regime—and we are left only with the problem of the field itself. The field coupled to a thermal reservoir is subject to dissipation but it may also experience noise through an incoherent pump mechanism. As a result, the dynamic process becomes irreversible and has to be described by a master equation, a master equation now for the field:

$$\frac{d\rho}{dt} = -\frac{i}{h}[H, \rho] + \left(\frac{d\rho}{dt}\right)_{\text{irr}} \tag{3.19}$$

The irreversible part of the time evolution is typically of the form

$$\left(\frac{d\rho}{dt}\right)_{\text{irr}} = \frac{\Gamma_1}{2}[b, \rho b^\dagger] + \frac{\Gamma_1}{2}[b\rho, b^\dagger] + \frac{\Gamma_2}{2}[b^\dagger, \rho b] + \frac{\Gamma_2}{2}[b^\dagger\rho, b] \tag{3.20}$$

where $\Gamma_1 = \Gamma(n_{\text{th}} + 1)$ and $\Gamma_2 = \Gamma n_{\text{th}}$. The reservoir is assumed to be in equilibrium at a certain temperature T, and the thermal population of the resonant reservoir mode is

$$n_{\text{th}} = [\exp(h\omega/kT) - 1]^{-1}$$

Γ is the rate of relaxation of $\langle b^\dagger b \rangle$ into the vacuum, and $b^\dagger$ and b are the creation and annihilation operators for a given field mode. For problems with more than a single excited mode, the master equation contains similar terms for every mode present. Expanding the statistical operator on the basis of the Fock states of the field, we may write

$$\rho(t) = \sum_{n,m=0}^{\infty} |n > \rho_{n,m}(t) < m| \tag{3.21}$$

and the dynamic equation for the density matrix elements can formally be written as

$$\frac{d\rho_{n,m}}{dt} = \sum_{i,j} \Lambda^{i,j}_{n,m} \rho_{i,j}(t) \tag{3.22}$$

The generator Λ of the time evolution summarizes the reversible, Hamiltonian contributions as well as the irreversible ones. In general this matrix equation is rather difficult to handle, and for realistic problems the diagonalization must be performed numerically on a truncated basis. However, the intuitive physical insight into a given problem provided by this matrix formalism is rather limited, and it is not immediately clear how to insert, for instance, simplifying assumptions that are consistent with the requirements of a master equation.

For that purpose, the quasi-probability representation developed by Glauber and Sudarshan in the form of the P-representation(7,8) is much more useful, since it crreates an intuitive picture for the physical process, as well as a formalism quite similar to those of classical statistical physics. It can be shown that ρ can always be represented by a c-number function $P(\alpha, \alpha^*, t)$ if we generalize the admissible function space and also allow for distributions such as the delta function and its derivatives:

$$\rho(t) = \int d^2\alpha \, |\alpha > P(\alpha, \alpha^*, t) < \alpha| \tag{2.23}$$

From the knowledge of $P(\alpha, \alpha^*, t)$ one can calculate all relevant ensemble averages, provided they have been arranged in normal order first:

$$\langle b^{\dagger n} b^m \rangle = \int \alpha^{*n} \alpha^m P(\alpha, \alpha^*, t) \, d^2\alpha \tag{3.24}$$

Instead of the matrix $\rho_{n,m}$ which evolves according to the matrix equation above (3.22), one can describe the process now by a real function of the continuous variables α and α^*, as in classical statistical mechanics. The master equation for ρ then turns into a partial differential equation for $P(\alpha, \alpha^*, t)$, which typically is of the following form:

$$\begin{aligned}\frac{\partial P(\alpha, \alpha^*, t)}{\partial t} = & -\frac{\partial}{\partial \alpha}(d-\chi\,|\alpha|^2)\,\alpha P(\alpha, \alpha^*, t) - \frac{\partial}{\partial \alpha^*}(d-\chi\,|\alpha|^2)\,\alpha^* P \\ & + \frac{1}{2} Q \frac{\partial^2}{\partial \alpha\, \partial \alpha^*} P(\alpha, \alpha^*, t) \\ & + \chi \left(\frac{\partial^2}{\partial \alpha^2} \alpha^2 + \frac{\partial^2}{\partial \alpha^{*2}} \alpha^{*2} + \cdots \right) P(\alpha, \alpha^*, t) \\ & + \cdots \frac{\partial^3}{\partial \alpha^3} \cdots \frac{\partial^3}{\partial \alpha^{*3}} \cdots + \cdots \end{aligned} \tag{3.25}$$

d represents an external control parameter, χ is the nonlinear coupling constant, and Q is the strength of additive noise. At a first look, the structure of this equation seems quite familiar, since to a certain extent it resembles the Fokker–Planck equation of classical statistics. At a closer look, however, we realize that the diffusion matrix associated with this equation is not necessarily positive semidefinitive (i.e., when $\chi \neq 0$) and the resulting equation is not of Fokker–Planck form. Also, the terms indicated by the dots in the last line, which stand for possible higher order derivatives, must be negligible in order to allow for such a statistical interpretation. Provided this is the case, the dynamics of a quantum statistical problem can be cast into the form of a purely classical stochastic process and the only trace of the inherent quantum nature left is the required ordering of the operators. The diffusion term, proportional to Q, can contain contributions that may be of classical or of quantum origin. If we choose a thermal reservoir, as in the example indicated above, Q is proportional to the thermal population $n_{\rm th}$, while in case of an incoherent pumping process, Q scales with $(n_{\rm th}+1)$ and remains finite even for vanishing temperatures. This demonstrates intuitively that classical fluctuations, as well as quantum noise, which is responsible for the 1 in $(n_{\rm th}+1)$, enter the formalism on quite equal footing and their influences merely add up.

This is quite different when we consider the other terms in the evolution equation, which are also of second order in the field derivatives (i.e., $Q=0$ and $\chi \neq 0$). In this case the equation above is by no means a Fokker–Planck equation and a corresponding classical stochastic process does not exist. We still may use the term noise, but now in a more abstract sense. This will become clearer by presenting two illustrative examples: a linear oscillator with gain, and a parametric process, where the "diffusive" terms originate from the reversible part of the dynamics.

A linear harmonic oscillator which experiences gain through the coupling to an incoherent source may be described by the following Fokker–Planck equation[9]:

$$\frac{\partial P(\alpha, \alpha^*, t)}{\partial t} = -\frac{\partial}{\partial \alpha} \Gamma \alpha P - \frac{\partial}{\partial \alpha^*} \Gamma \alpha^* P + 2\Gamma \frac{\partial^2}{\partial \alpha\, \partial \alpha^*} P(\alpha, \alpha^*, t) \quad (3.26)$$

where Γ represents gain, which is inevitably accompanied by quantum noise. The solution for this process is obtained straightforwardly, since it is virtually identical to the celebrated Ornstein–Uhlenbeck process. When the trajectories initiate from the vacuum at $t = 0$, then the average field amplitudes $\langle \alpha \rangle$ and $\langle \alpha^* \rangle$ remain zero for all times, while the photon number builds up from zero, intuitively speaking, due to the action of the intrinsic quantum noise:

$$\langle b^\dagger(t)\, b(t) \rangle = \exp(\Gamma t) - 1 \quad (3.27)$$

In the short-time limit we find

$$\langle b^\dagger(t)\, b(t) \rangle = \Gamma t \quad (3.28)$$

This is quite analogues to the well-known result for Brownian motion, where the square of the distance $x^2(t)$ from an initial position grows linearly in time. This clearly demonstrates that quantum noise can behave qualitatively similar to classical noise, resulting in a diffusive motion.

Another quite simple example will demonstrate that this is not always the case, and quantum noise can behave quite differently. The linearized model for the generation of a subharmonic field is given by the following Hamiltonian:

$$H = \hbar\omega b^\dagger b + ig\hbar(b^2 e^{2i\omega t} - b^{\dagger 2} e^{-2i\omega t}) \quad (3.29)$$

where ω is the frequency of the subharmonic field and g is the nonlinear susceptibility or the coupling constant. In the rotating frame the explicit time dependence can be eliminated. In order to make the model a little more realistic, we add the coupling to a zero-temperature heat bath, which introduces dissipation, but no additional noise. The corresponding master equation for $P(\alpha, \alpha^*, t)$ can be written in the form

$$\partial P(\alpha, \alpha^*, t) = -\frac{\partial}{\partial \alpha}(-\gamma\alpha + g\alpha^*) P - \frac{\partial}{\partial \alpha^*}(-\gamma\alpha^* + g\alpha) P + g\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}}\right) P(\alpha, \alpha^*, t) \quad (3.30)$$

γ is the damping constant and g is the nonlinear susceptibility. This is obviously not a Fokker–Planck equation and therefore we cannot expect to obtain a diffusive motion as in the previous case. A formal solution of this equation, however, can be found quite easily, involving delta functions and their derivatives. Starting in the vacuum state at the initial time $t=0$, we find with $\alpha = x + iy$

$$P(x, y, t) = N \exp[+a(t)\, x^2] \exp\left[-b(t) \frac{\partial^2}{\partial y^2}\right] \delta(y) \tag{3.31}$$

where N,

$$N = \frac{(\gamma - 2g)^{1/2}}{\pi g} \frac{(1 - e^{(2g-\gamma)t})^{-1}}{2}$$

is the normalization factor and $a(t)$ and $b(t)$ are given by

$$a(t) = \frac{\gamma - 2g}{g(1 - e^{(2g-\gamma)t})}$$

$$b(t) = \frac{1}{4} \frac{g}{\gamma + 2g} (1 - e^{-(2g+\gamma)t}) \tag{3.32}$$

This is only a formal result, but nevertheless it is a very useful tool for calculating transient moments. While the first moments $\langle \alpha \rangle$ or $\langle \alpha^* \rangle$ remain zero again, the second moments, such as the field intensity, experience the instability. They are triggered by the quantum fluctuations, and evolve in time according to

$$\langle x(t)^2 \rangle = \frac{1}{2} \frac{g}{\gamma - 2g} (1 - e^{(2g-\gamma)t})$$

$$\langle y(t)^2 \rangle = -\frac{1}{2} \frac{g}{\gamma + 2g} (1 - e^{-(2g+\gamma)t}) \tag{3.33}$$

One should notice the rather unexpected result that the second moment of y, where $y = \mathrm{Im}(\alpha)$, becomes negative. However, since $[\mathrm{Im}(\alpha)]^2$ is not an observable, this is not in conflict with experimental results. Observable, for instance, is the intensity of the field $\langle b^\dagger(t)\, b(t) \rangle = \langle x^2(t) + y^2(t) \rangle$, which, for short times, as can be shown, does not feel the effects of dissipation, and therefore has to exhibit a time-reversal dynamics. This is indeed the case in the present example, and up to second order in t we find

$$\langle b^\dagger(t)\, b(t) \rangle = g^2 t^2 + O(t^3) \tag{3.34}$$

Obviously, this result is possible only since the diffusive linear term $\rightarrow t$ which appears in $\langle x^2 \rangle$ is canceled by an indentical term originating form $\langle y^2 \rangle$ but with the opposite sign. Thus, the cancellation is only possible because $\langle y^2 \rangle$ is not positive definite.

When comparing these two results, it is obvious that quantum noise can act in various ways. It can appear quite similar to classical noise and result in a purely random and diffusive behavior as demonstrated in the first example, or it can behave rather different from a classical stochastic process, by retaining the time-reversal symmetry of the underlying Hamiltonian dynamics. Common in both cases is the fact that either form of quantum noise has the ability to trigger the onset of time evolution when, in the classical limit, we have an unstable stationary point. Among the present examples, the latter one was rather peculiar, since there did not exist a positive probability density, but only a quasiprobability expressed in delta functions.

When choosing a positive distribution function $P(\alpha, \alpha^*, t)$ at the initial time $t=0$, the function will remain positive for all later times in the first example, and all the classical inequalities (3.15)–(3.18) will still hold in the quantum case. In the second example, however, the distribution function assumed negative values in the course of time and consequently these inequalities can be violated by such processes. In general we find that:

(i) The function

$$G_2(t) = \int (\alpha\alpha^* - \langle \alpha\alpha^* \rangle)^2 \, P(\alpha, \alpha^*, t) \, d^2\alpha \tag{3.35}$$

is not necessarily positive and antibunching[10–13] becomes possible.

(ii) The photon counting probability

$$W(n, T) = \int \frac{1}{n!} (\eta\alpha\alpha^* T)^n \exp(-\eta\alpha\alpha^* T) \, P_0(\alpha, \alpha^*) \, d^2\alpha \tag{3.36}$$

can be narrower than the Poisson distribution,[13–15] since P_0 is not restricted to positive values. P_0 is the stationary distribution, and for simplifying the result, we have assumed that T is short compared to the correlation time of the field, quite similar to the assumption used for the classical example above Eq. (3.16).

(iii) The variance

$$\langle \Delta x^2(t) \rangle = \frac{1}{4} \left[1 + \int (\alpha + \alpha^*)^2 \, P(\alpha, \alpha^*, t) \, d^2\alpha \right] \tag{3.37}$$

is not necessarily larger than 1/4 and squeezing[16–18] becomes possible.

In general, classical and quantum noise can be present simultaneously. A typical example would be the previous parametric process, but now interacting with a finite-temperature heat bath in addition. This results in a competition between classical and nonclassical effects. When the first ones dominate, classical behavior will be observed, and consequently there exists a regular, positive probability density for

$$g < 2\,\frac{n_{\text{th}}}{1 + n_{\text{th}}}\,\gamma$$

On the other hand, when this inequality is violated, a positive distribution may only be prepared initially, but in course of time it will develop regions of negative values and nonclassical behavior previals.

3.3. The Positive *P*-Representation

With the term noise or fluctuations we always associate an irregular temporal behaviour, typically something like the classical Brownian motion. In quantum mechanics the term fluctuations arises, since there all physical observables are expressed in terms of ensemble averages of operators and the variances of such quantities in general do not vanish. This is reminiscent of classical statistics and one might be tempted think of quantum noise as some kind of random process. This, however, is not the case and we will see that we cannot associate a Langevin equation with every quantum dynamical problem.

The use of the quasiprobability concept has the advantage that it uses the same language for both cases and it allows one to utilize the methods developed for classical stochastic processes also in the quantum case. For a nonlinear dynamic problem, the drift and the diffusion coefficients become nonlinear, and a solution of the partial differential equation becomes increasingly more complicated, the more variables that are involved. However, in cases where the dynamics follows from a genuine classical or quantum Fokker–Planck equation, there exists an alternative concept, the statistically equivalent Langevin equation. Representing only a set of ordinary differential equations, they are easier to handle and at least can be simulated numerically without major difficulties. This may not be the most elegant way to solve a statistical process, but it is nevertheless useful for gaining a first insight into a new phenomenon. This remedy, however, cannot cure the problem in the case where the quantum fluctuations display their typical nonclassical behavior. In such a case, when the Fokker–Planck concept breaks down and variances become negative, due to the negative regions in the distribution function, a corresponding Langevin equation does not exist in principle.

Drummond and Gardiner[19,20] nevertheless have found a way out of this dilemma, by defining an alternative quasiprobability concept on an extended phase space with a distribution function that remains positive for all times. This enables one to find a Fokker–Planck equation for all quantum processes, irrespective of the form of the diffusion coefficient, provided that the equation for $P(\alpha^*, \alpha, t)$ can be approximated by a second-order differential equation. The new quasiprobability, the so-called positive P-representation, is defined on an enlarged phase space and depends on (α, α^*) and an additional pair of variables $(\beta. \beta^*)$ for each mode of the field:

$$P(\alpha, \alpha^*, \beta, \beta^*, t) \geqslant 0 \tag{3.38}$$

Ensemble averages of normally ordered operator products are then calculated in the following way:

$$\langle (b^\dagger)^n (b)^m \rangle_t = \iint d^2\alpha\, d^2\beta (\beta)^n (\alpha)^m P(\alpha, \beta, t) \tag{3.39}$$

It can be shown that the evolution equation for $P(\alpha, \beta, t)$ is easily found, provided the equation of motion for $P(\alpha, \alpha^*, t)$ is known, by merely replacing

$$\alpha \to \alpha \qquad \text{and} \qquad \alpha^* \to \beta \tag{3.40}$$

For any given statistical operator one can prove that the positive P-representation does exist, but it is not unique, as one may expect from an expansion in an overcomplete set of basis functions.

The usefulness of this concept may most clearly be demonstrated by the use of an explicit example, like the previous problem of a linearized parametric process, where the Glauber representation yields a complex solution in terms of deltafunctions and their derivatives. By replacing α^* by β we find the following equation for $P(\alpha, \beta, t)$:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \alpha}(-\gamma\alpha + g\beta) P - \frac{\partial}{\partial \beta}(-\gamma\beta + g\alpha) P + g\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right) P(\alpha, \beta, t) \tag{3.41}$$

Due to the analytic structure of this method, we can identify the derivatives either by

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial x} \qquad \text{and} \qquad \frac{\partial}{\partial \beta} = \frac{\partial}{\partial u} \tag{3.42a}$$

or by

$$\frac{\partial}{\partial \alpha}=\frac{\partial}{\partial iy} \quad \text{and} \quad \frac{\partial}{\partial \beta}=\frac{\partial}{\partial iv} \tag{3.42b}$$

where $\alpha = x + iy$ and $\beta = u + iv$. Thereby the diffusion matrix in this simple example can explicitly be made positive semidefinite by choosing

$$\frac{\partial^2}{\partial \alpha^2}+\frac{\partial^2}{\partial \beta^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial u^2} \tag{3.43}$$

The Langevin process corresponding to this Fokker–Planck equation is given by the following set of stochastic differential equations:

$$\begin{aligned} \dot{x} &= -\gamma x + gu + \xi_1(t) \quad &\text{and}& \quad \dot{y} = -\gamma y + gv \\ \dot{u} &= -\gamma u + gx + \xi_2(t) \quad &\text{and}& \quad \dot{v} = -\gamma v + gy \end{aligned} \tag{3.44}$$

The fluctuating forces ξ_1 and ξ_2 represent two independent Gaussian white noise forces:

$$\langle \xi_i(t)\, \xi_j(t') \rangle = \delta_{ij}\, g^{1/2} \delta(t-t') \tag{3.45}$$

If the dynamics would be nonlinear, this could be the starting point for a numerical simulation. In the present case, however, the problem can be solved quite easily and we can find the probability distribution for all times in closed analytical form.

As a first step we may be interested in the stationary solution of Eq. (3.41), which can be shown to assume the following form:

$$P(\alpha, \beta, t \to \infty) = P_0 = \frac{1}{\pi}(\lambda-1)^{1/2} \exp[-\lambda(x^2+u^2)+2xu]\, \delta(y)\, \delta(v) \tag{3.46}$$

where $\lambda = \gamma/g > 1$ is the only relevant parameter in the asymptotic limit. This inequality guarantees the validity of the linearized model, by restricting it to the subthreshold region. The stationary moments are easily found by simple quadratures:

The intensity of the field

$$\langle b^\dagger b \rangle = \frac{1}{2}\frac{\lambda}{\lambda^2-1} \tag{3.47}$$

The variances of the field

$$\left\langle\left(\frac{b^\dagger + b}{2}\right)^2\right\rangle = \frac{1}{4}\frac{\lambda}{\lambda - 1} \tag{3.48}$$

$$\left\langle\left(\frac{b^\dagger - b}{2i}\right)^2\right\rangle = \frac{1}{4}\frac{\lambda}{\lambda + 1} \tag{3.49}$$

It is obvious that the variances of the real and imaginary parts of the field no longer satisfy the inequalities set by the classical dynamics and sqeezing becomes possible. The time-dependent solution is found by straightforward integration. It describes the transient relaxation toward the equilibrium state. Starting from the vacuum at $t=0$, we find

$$P(x, y, u, v, t) = N \exp[-D(x^2 + u^2) - 2Fxu]\, \delta(y)\, \delta(v) \tag{3.50}$$

where

$$N = \frac{1}{\pi}\frac{(\lambda^2 - 1)^{1/2}}{[1 + e^{-2\gamma t} - 2\cosh(gt)\, e^{-\gamma t}]^{1/2}}$$

$$D = \frac{\lambda + e^{-\gamma t}[\lambda \cosh(gt) + \sinh(gt]}{1 + e^{-2\gamma t} - 2e^{-\gamma t}\cosh(gt)}$$

$$F = \frac{1 - e^{-\gamma t}[\lambda \sinh(gt) + \cosh(gt]}{1 + e^{-2\gamma t} - 2e^{-\gamma t}\cosh(gt)}$$

With this brief review of the qualitative properties of quantum fluctuations and the mathematical tools available to treat such processes that are dominated by quantum noise, we close this section and turn to explicit and quantitative examples in the next section.

4. EXAMPLES OF QUANTUM OPTICAL PROCESSES WITH NOISE

In the previous sections we have presented a number of basic concepts of quantum optics as much as they pertain to questions of noise. We have also discussed the mathematical framework for describing and solving the corresponding statistical equations. In the present section we will now make use of these concepts and discuss several physical examples in more detail, where we emphasize especially the role of fluctuations. We will not go through elaborate mathematical derivations, but only motivate the structure of the physical models and their noise properties. Then we turn directly to the explicit solutions and discuss their physical interpretation and consequences.

4.1. The Laser with Additive and Multiplicative Noise

The most important problem in quantum optics is unquestionably the laser. The theoretical description is well established and in good agreement with experiment. Nevertheless, new experiments have recently revived interest in alternative theoretical models by taking into account fluctuations different from the pure spontaneous emission noise. Considering a certain randomness in the dissipative couplings leads to a laser model with multiplicative noise. This concept was not widely used in the past, and it was mostly the example of the dye-laser that has demonstrated that multiplicative processes really occur in nature. In the meanwhile, a great deal of theoretical as well as experimental work has appeared in this field. The "classical" laser model can be understood as the generic nonlinear process driven by additive noise. The laser models with loss or gain fluctuations can serve as the standard process with multiplicative noise, expecially since an exact analytical solution exists for the stationary as well as the dynamic equations.

4.1.1. The Laser with Spontaneous Emission Noise. When spontaneous emission noise dominates, we find the "classical" laser model, treated in great detail by many authors. For this reason we will only briefly list here the main results, serving as a reference for comparison with the additional laser models we discuss below. By eliminating the rapidly relaxing atomic degrees of freedom in favor of a slow-field mode, we find the following stochastic model:

$$dE/dt = (d - |E|^2)\, E + \xi(t) \tag{4.1}$$

where $\mathrm{Re}(d)$ represents the balance of gain and loss, and $\mathrm{Im}(d)$ is the detuning between atoms and field, which we will disregard in the following. $\xi(t)$ describes the stochastic influence of spontaneous emission in Markov approximation:

$$\langle \xi(t)\, \xi(t') \rangle = Q\delta(t - t') \tag{4.2}$$

The corresponding Fokker–Planck equation is the celebrated "laser equation"[21,22]:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial E}(d - |E|^2)\, EP - \frac{\partial}{\partial E^*}(d^* - |E|^2)\, E^* P + \frac{Q}{2}\frac{\partial^2}{\partial E\, \partial E^*} P \tag{4.3}$$

where $P = P(E, E^*, t)$. The stationary solution for this process can be given in closed form, since the condition of detailed balance happens to be satisfied:

$$P_0(E, E^*) = N \exp\left[\frac{4}{Q}\left(d\,|E|^2 - \frac{1}{2}\,|E|^4\right)\right] \tag{4.4}$$

N is the normalization constant, containing mainly for the error function. This analytic result allows one to calculate all stationary moments and the classical photon counting statistics explicitly.

(i) The field intensity $\langle |E|^2 \rangle$ grows slowly from very small values below threshold, $d < 0$, and then smoothly turns over into a linear rise well above threshold, $d > 0$:

$$\begin{aligned} \langle |E|^2 \rangle &= Q/d \qquad \text{for} \quad d \ll -1 \\ \langle |E|^2 \rangle &= 2d/Q \qquad \text{for} \quad d \gg +1 \end{aligned} \tag{4.5}$$

(ii) A measure for the randomness of the field intensity $I = |E|^2$ is the relative variance:

$$\frac{\langle \Delta I^2 \rangle}{\langle I \rangle^2} = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} \tag{4.6}$$

which starts from $\langle \Delta I^2 \rangle = 1$ well below theshold, indicating a violently fluctuating field, until it falls off like $\langle \Delta I^2 \rangle = Q^2/2d^2$ well above threshold, where the nonlinearity dominates and stabilizes the laser amplitude.

(iii) The photon counting statics can be calculated directly from the stationary distribution in the limit, where the correlation time of the field is long compared with the experimental observation time T:

$$W(n, T) = \int_0^\infty \frac{1}{n!} (\eta I T)^n \exp(-\eta I T)\, P_0(I)\, dI \tag{4.7}$$

The fluctuations of the photon number are easily related to the moments of the field intensity:

$$\langle n \rangle = \eta T \langle I \rangle \qquad \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2} = \langle \Delta I^2 \rangle + \frac{1}{\langle n \rangle} \tag{4.8}$$

The second formula clearly demonstrates how the counting statistics changes from the equilibrium distribution of the Bose–Einstein statistics well below theshold, $d \ll -1$, $\langle \Delta I^2 \rangle = 1$, into a Poisson process when the amplitude is stabilized well above threshold, $d \gg +1$, $\langle \Delta I^2 \rangle = 0$.

The essential dynamic properties of the laser process are contained in the field- and the intensity-correlation functions:

$$G_1(t) = \langle E^*(t)\, E(0) \rangle, \qquad G_2(t) = \langle I(t)\, I(0) \rangle \tag{4.9}$$

which in the threshold region are dominated by a single exponential:

$$G_1(t) \simeq \exp(-\lambda_0^1 t), \qquad G_2(t) \simeq \exp(-\lambda_1^0 t) \tag{4.10}$$

where

$$\lambda_0^1 \simeq -d \text{ for } d \ll -1 \quad \text{and} \quad \lambda_0^1 \simeq \frac{1}{d} \text{ for } d \gg +1$$

$$\lambda_1^0 \simeq -d \text{ for } d \ll -1 \text{and} \quad \lambda_1^0 \simeq 2d \text{ for } d \gg +1$$

These results are illustrated qualitatively in Figs. 1–5.

4.1.2. Laser with Loss Fluctuations. The most interesting regime of laser operation in theory is the immediate neighborhood of the threshold. There, all attempts to linearize the dynamics are bound to fail and the problem exhibits its full nonlinear behavior. This is quite obvious, since in the threshold regime neither the lasing nor the random spontaneous emission mode is globally stable and the competition persists. The traditional laser model has been found to be in excellent agreement with experiments close to threshold and for a long time there was no reason to generalize the model. Experiments on dye-lasers,[23,24] however, suddenly revived the search for alternative laser models, since the discrepancy of those new results with theory was quite dramatic. When interpreting the experimental results, it became clear soon that it is the additive noise assumption that poorly models the dominant fluctuations in the previous theories and that multiplicative noise should be responsible for the unusual behavior.

One possibility for multiplicative noise is a certain randomness in the

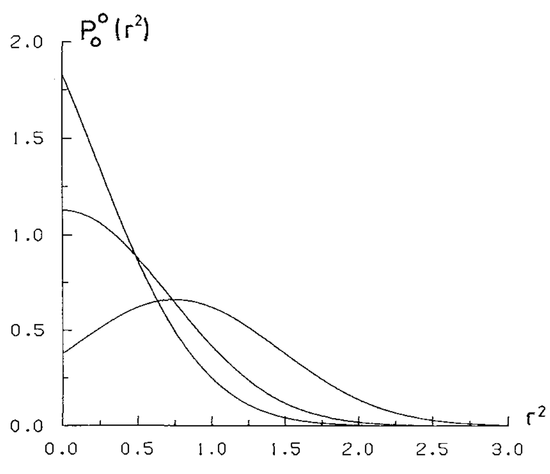


Fig. 1. Steady-state distribution of the laser below ($d = -1$), at ($d = 0$), and above threshold ($d = 1.5$).

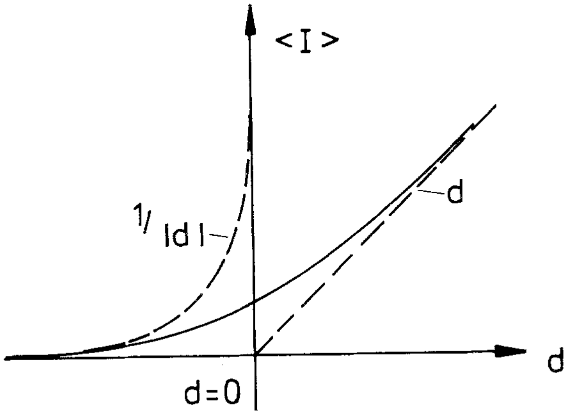


Fig. 2. Average intensity as a function of the pump parameter *d*.

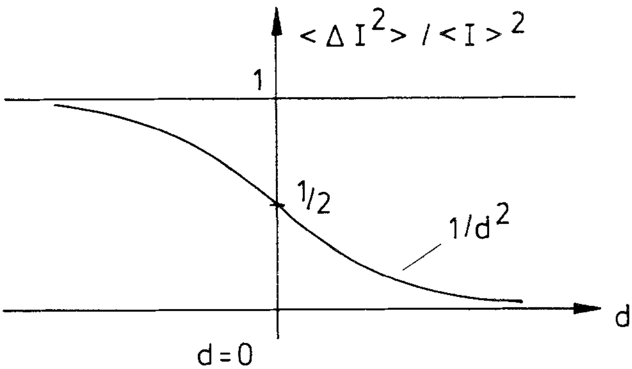


Fig. 3. Relative variance of the intensity as a function of the pump parameter.

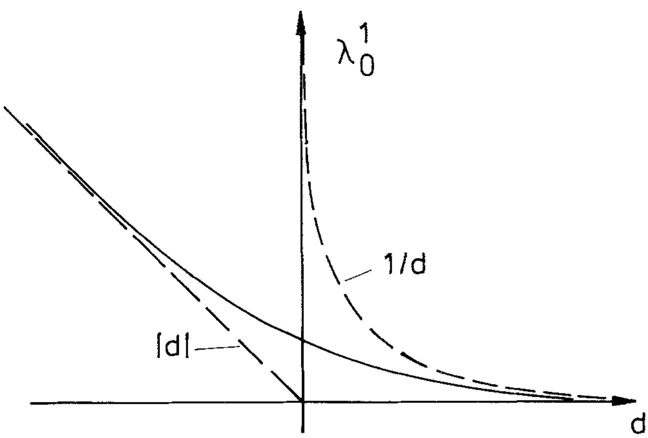


Fig. 4. Smallest eigenvalue of the Fokker–Planck equation describing phase fluctuations.

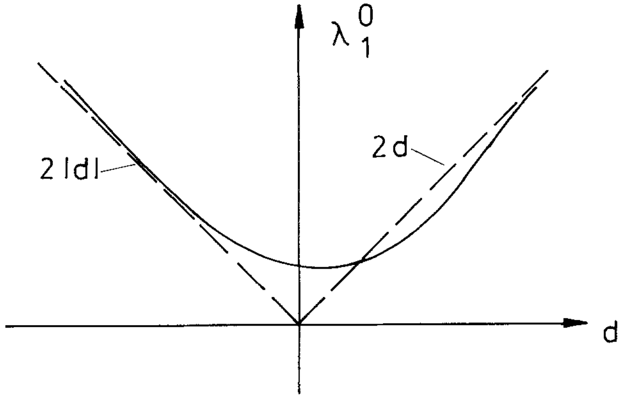


Fig. 5. Smallest eigenvalue of the Fokker–Planck equation describing intensity fluctuations.

dissipative mechanisms, which causes the linear gain parameter d to fluctuate about a certain average value, leading to the following model[25]:

$$\frac{dE}{dt} = (d - |E|^2)\, E + \xi_1(t) + E\xi_2(t) \tag{4.11}$$

where ξ_1 describes the ever-present spontaneous emission and ξ_2 the noise associated with the loss mechanisms. To emphasize the role of $\xi_2(t)$, we assume that it is the dominant source of noise and we neglect for the moment spontaneous emission entirely. In order to construct the simplest model possible, we further assume that the real and the imaginary part of ξ_2 are represented by two independent Gaussian white noise fources of equal strength. The corresponding Fokker–Planck equation for $E = r\exp(i\phi)$ then reads[26,27]:

$$\frac{\partial P}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(dr - r^3 + \frac{Q}{2}\frac{\partial}{\partial r} r^2 \right) P \right] + \frac{Q}{2}\frac{\partial^2}{\partial \phi^2} P(r, \phi, t) \tag{4.12}$$

The phase invariance of the problem suggests the following eigenfunction ansatz:

$$P(r, \phi, t) = P_n^m(r) \exp(im\phi) \exp(-\lambda_n^m t) \tag{4.13}$$

The steady-state solution $P_0^0(r)$ is easily obtained and we find for $d > 0$

$$P_0^0(r) = \frac{1}{\pi} Q^{-d/Q} \Gamma^{-1}\left(\frac{d}{Q}\right) r^{-2+2d/Q} \exp\left(-\frac{1}{Q} r^2\right) \tag{4.14}$$

The most remarkable feature of this process is the fact that the dynamic eigenvalue problem can also be solved analytically, in contrast to the "classical" laser process, which has never been solved in analytical form. We find

$$P_n^m(r, \phi) = Nr^{-2+2d/Q-2n} \exp\left(-\frac{1}{Q}r^2\right) \exp(im\phi)$$

$$\times {}_1F_1\left(-n, \frac{d}{Q} - 2n + 1, \frac{1}{Q}r^2\right) \tag{4.15}$$

with the corresponding eigenvalues

$$\lambda_n^m = Q\left[\frac{1}{2}m^2 + 2n\left(\frac{d}{Q} - n\right)\right] \tag{4.16}$$

N is the normalization constant. The integrability condition for the discrete spectrum can only be satisfied for

$$d/Q > 2n_0$$

where n_0 is the largest natural number that satisfies the inequality above. $n_0 + 1$ is the number of discrete eigenvalues, including the steady state. Above the discrete spectrum of relaxation rates λ_n^m there exists a continuum of eigenvalues λ, very analogous to the quantum mechanical problem of a one-dimensional potential well. The eigenfunctions that generate the continuous branch of the spectrum are found in the usual way by relaxing the condition of square integrability. The properties of the continuum may even dominate the dynamic behavior, when the laser is operated close to threshold, i.e., $d/Q < 2$, where there exists no discrete eigenvalue besides the steady state.

With these explicit results it is possible to determine all relevant physical properties in analytical form, such as the stationary moments and correlation functions. Experimentally, it is much easier to measure the field intensity and its temporal correlations. Therefore it is useful to formulate the corresponding process for $I = |E|^2$ directly:

$$\frac{dI}{dt} = (\Gamma - \kappa)\, I - \Gamma I^2 + I\xi(t) \tag{4.17}$$

and

$$\frac{\partial P(I)}{\partial t} = -\frac{\partial}{\partial I}(\Gamma - \kappa - \Gamma I)\, IP + \frac{\partial}{\partial I} I \frac{\partial}{\partial I} IP \tag{4.18}$$

where Γ is the gain and κ the loss parameter. This form is also helpful for comparison with the results we discuss below. In order to have a consistent notation, we have not normalized the intensity in these two equations. In this notation, the average intensity of the laser is given by the simple relation

$$\langle I \rangle = (\Gamma - \kappa)/\Gamma$$

while the relative variance of the intensity assumes the following form:

$$\langle \Delta I^2 \rangle / \langle I \rangle^2 = 1/(\Gamma - \kappa)$$

These results are limited to the regime $d = \Gamma - \kappa > 0$, since for $d < 0$ the additive spontaneous emission noise can no longer be neglected, as it is the only remaining noise, which prevents the system from collapsing into the vaccum $E = 0$. The variance of the present process is not bounded from above and can grow to arbitrary large values close to threshold. The actual divergence is prevented by the addition of weak additive noise. This behavior is in satisfactory agreement with the experimental observations made on the dye-laser system Fig. 6 and it is in strong contrast to the traditional laser theory.

With the explicit form of the stationary distribution $P_0(I)$

$$P_0(I) = \Gamma^{\Gamma - \kappa} / \Gamma_0(\Gamma - \kappa)\, I^{\Gamma - \kappa - 1} e^{-\Gamma I}$$

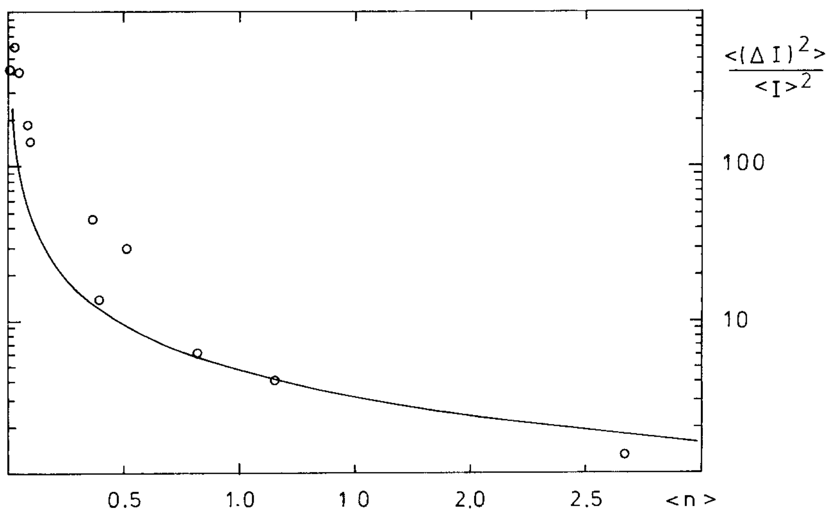


Fig. 6. Relative variance of the intensity. Comparison of—theoretical result with (○) experiment.

we can also obtain the photon counting statistics $W(n, T)$

$$W(n, T) = (\eta T)^n \frac{1}{n!} \frac{\Gamma^{n-\kappa}}{(\Gamma+\eta T)^{n+\Gamma-\kappa}} \frac{\Gamma_0(n+\Gamma-\kappa)}{\Gamma_0(\Gamma-\kappa)}$$

which is compared with experiments in Fig. 7. The Γ_0 is the gamma function, not to be mixed up with the Γ parameter of the model. Here, too, the qualitative agreement between theory and experiment is rather satisfactory.

Also, the experimentally observed relaxation of intensity fluctuations indicates that the classical laser model does not apply in the case of the dye-laser. The classical laser model predicts an almost exponential decay of the two-photon correlation function. In the experiment close to threshold, the intensity correlation function is clearly not governed by a single decay rate. However, this property is an immediate consequence of the present

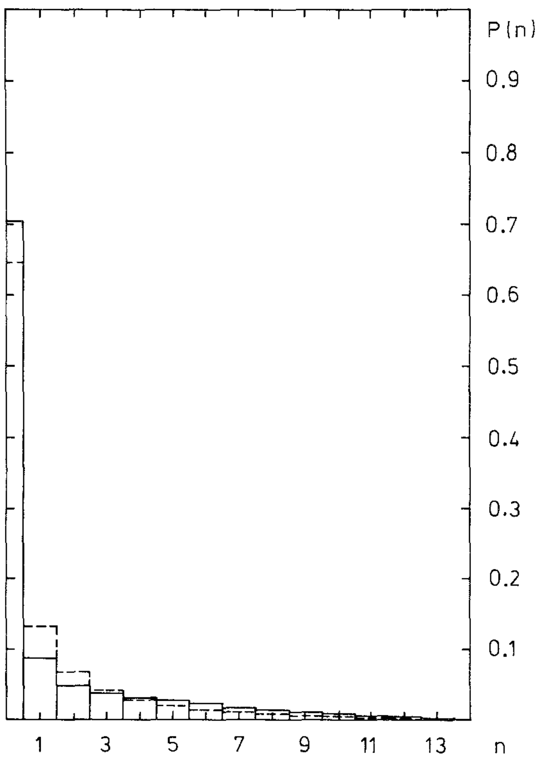


Fig. 7. Photon counting statistics $W(n, T)$ for the laser with loss noise. Comparison of (—) theory with (– –) experiment.

multiplicative noise model. Since close to threshold, i.e., $d/2Q < 1$, the spectrum of relaxation rates becomes purely continuous, there is no *a priori* reason for observing a simple exponential decay. The corresponding correlation function can be calculated analytically up to a final quadrature with the use of the continuous eigenfunctions. The theoretical result is plotted and compared with experiment in Fig. 8.

This example demonstrates that multiplicative noise is not only of interest from a mathematical or academic point of view, but that it is a relevant source of noise appearing naturally in physical systems.

4.1.3. Laser with Gain Fluctuations. The physical motivation behind the previous model was based on the idea that the control parameter d was not entirely under external control, but subject to fluctuations $d \to d + \xi(t)$. Physically speaking, d represents the balance of gain and loss and one might attribute these fluctuations either to the pumping or to the damping mechanism. At a closer look, however, it is clear that this form of noise can only model the randomness of dissipation. The gain factor not only enters d, but also the parameters that scale the nonlinearity of the laser model. However, when considering the possible physical origin of such external noise, then fluctuations in the pump parameter seem to be the most likely source of randomness. They may result from fluctuations in the dye-jet or from a noisy pump laser. We will present a model with such noise properties here.

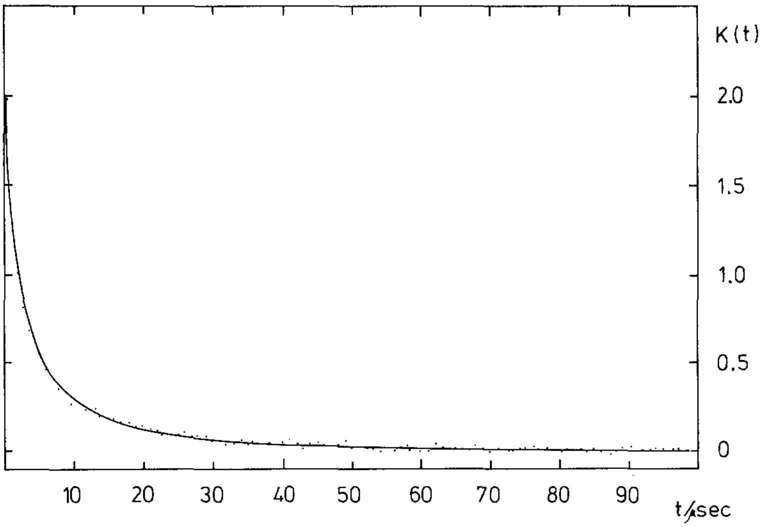


Fig. 8. Normalized intensity correlation function. Comparison of (—) theory with (·) experiment.

In terms of the field intensity, the model that describes pump fluctuations is of the following form[28]:

$$\frac{dI}{dt} = (\Gamma - \kappa) I - \Gamma I^2 + I(1 - I)\, \xi(t) \tag{4.19}$$

where $\xi(t)$ is a source of Gaussian white noise:

$$\langle \xi(t)\, \xi(t=0) \rangle = \delta(t) \tag{4.20}$$

κ is the cavity damping constant and Γ the gain factor, i.e., $\Gamma - \kappa = 0$ determines the laser threshold. The Fokker–Planck equation corresponding to this Langevin process is given by

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial I}(\Gamma - \kappa - \Gamma I)\, IP + \frac{\partial}{\partial I}(1 - I)\frac{\partial}{\partial I}(1 - I)\, P(I, t) \tag{4.21}$$

The stationary solution of this model is found by direct integration:

$$P_0(I) = \frac{\kappa^{\Gamma - \kappa}}{\Gamma_0(\Gamma - \kappa)} \frac{I^{\Gamma - \kappa - 1}}{(1 - I)^{\Gamma - \kappa + 1}}\, e^{-\kappa I/(1 - I)} \tag{4.22}$$

where we had to use the symbol Γ_0 again for the gamma function. The dynamic problem can also be solved analytically and we find again a discrete and a continuous branch of the spectrum. The discrete eigenvalue problem is solved by

$$P_n(I) = N_n I^{\Gamma - \kappa - n - 1}(1 - I)^{-\Gamma + \kappa + n - 1}\, L_n^{\Gamma - \kappa - 2n}\left(\frac{\kappa I}{1 - I}\right) e^{-\kappa I/(1 - I)} \tag{4.23}$$

with

$$N_n = \frac{1}{2}\kappa^{2(\Gamma - \kappa - n)} \frac{n!\,(\Gamma - \kappa - 2n)}{\Gamma_0(\Gamma - \kappa)\,\Gamma_0(\Gamma - \kappa + 1 - n)} \tag{4.24}$$

and the eigenvalues are given by

$$\lambda_n = n(\Gamma - \kappa - n) \qquad \text{with} \quad 2n < \Gamma - \kappa \tag{4.25}$$

L_n^m are the generalized Laguerre polynomials. Besides this finite number of discrete eigenvalues there is a continuum, characterized by a parameter s:

$$P_s(I) = N_s^{1/2} I^{(\Gamma - \kappa - 3)/2}(1 - I)^{(-\Gamma + \kappa - 1)} \times W_{1/2 + (\Gamma - \kappa)/2,\, is/2}\left(\frac{\kappa I}{1 - I}\right) e^{-(\kappa/2)[I/(1 - I)]} \tag{4.26}$$

where $W_n(x)$ is the Whittaker function and

$$N_s = \frac{1}{(2\pi)^2 \kappa^{\Gamma-\kappa-1}} s \frac{\sinh(\pi s)}{\Gamma_0(\Gamma-\kappa)} \left| \Gamma_0 \left(-\frac{\Gamma}{2} + \frac{\kappa}{2} + \frac{is}{2} \right) \right|^2 \tag{4.27}$$

The eigenvalues are given by

$$\lambda_s = \frac{1}{4} [(\Gamma-\kappa)^2 + s^2] \qquad \text{for} \quad s > 0 \tag{4.28}$$

With these explicit results we can calculate again the stationary as well as the dynamic properties. The stationary moment of the intensity I is

$$\langle I \rangle = (\Gamma-\kappa)\, \kappa^{\Gamma-\kappa} e^{\kappa} \gamma(-\Gamma+\kappa, \kappa) \tag{4.29}$$

where $\gamma(x, y)$ is the incomplete gamma function. The second moment becomes rather unwieldy and we only give here the asymptotic result for $\kappa \gg \Gamma - \kappa$:

$$\langle I^2 \rangle = \frac{\Gamma-\kappa}{\kappa} \frac{\Gamma-\kappa+1}{\kappa} \left(1 - \frac{2}{\kappa} (\Gamma-\kappa+2) \pm \cdots \right) \tag{4.30}$$

In this limit we find for the relative variance of the intensity fluctuations the following result:

$$\frac{\langle \Delta I^2 \rangle}{\langle I \rangle^2} = \frac{1}{\Gamma-\kappa} \left[1 - \frac{2}{\kappa} (\Gamma-\kappa+1) \right] \tag{4.31}$$

which obviously generalizes the previous expression. The intensity correlation function in the regime of the discrete spectrum can be found analytically, but it is rather complicated in form. For large times the correlation function is dominated by the slowest eigenvalue:

$$\langle I(t)\, I(t=0) \rangle = \langle I \rangle^2 + A \exp(-\lambda_1 t), \qquad \lambda_1 = \Gamma - \kappa - 1 \tag{4.32}$$

where A is a rather involved factor of the order of $\langle I^2 \rangle - \langle I \rangle^2$. Very close to theshold, where the spectrum is entirely continuous, the correlation function can be calculated analytically for $\Gamma - \kappa \ll \kappa$ up to a final integral over the eigenvalues s. The results of the numerical integration are shown together with the experimental curves in Fig. 9 and the agreement is satisfactory. However, since the differences in the theoretical results, from model to model, are not really substantial at the moment, the fluctuations cannot be traced back to any specific physical source with certainty i.e. to gain or loss fluctuations. Nevertheless, one expects that noise in the pump field is responsible for this behavior and that the present model provides the most realistic description.

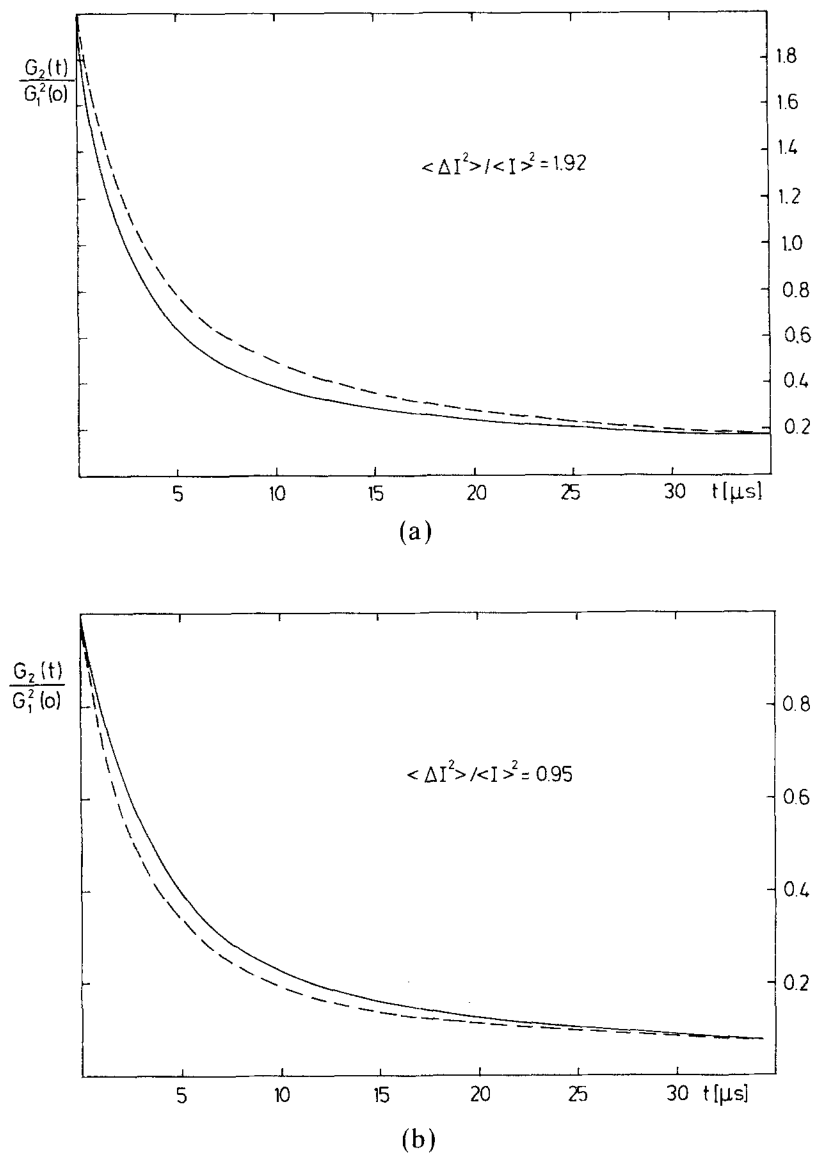
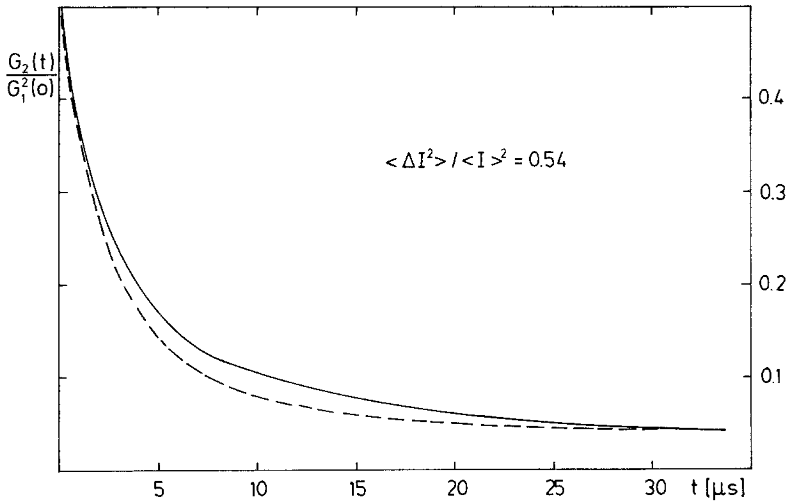


Fig. 9. Normalized intensity correlation function for a model with gain noise. Comparison of (—) theory and (– –) experiment. The distance from threshold increases from (a) to (c). The control parameter $\Gamma - \kappa$ was taken to be 0.52, 1.05 and 1.85 in (a), (b) and (c) respectively.



(c)

Fig. 9 *(continued)*

4.1.4. Laser with Multiplicative Nonwhite Noise. We have been looking for possible generalizations of the standard laser model, in order to describe the fluctuation behavior of the dye-laser. Since the physical source of noise is not known precisely, it is impossible to judge the validity of the white noise assumption at the moment. It is of interest therefore to investigate the properties of a laser model with nonwhite multiplicative noise.

A standard generalization of white noise models, which does not leave the framework of Markov processes, introduces an auxiliary variable $y(t)$. This variable has no immediate physical meaning, nor can it be measured directly. $y(t)$ acts as the noise source undergoing the traditional dynamics of an Ornstein–Uhlenbeck process,[29,30]

$$\begin{aligned} \frac{dI}{dt} &= (d-I)\, I + \frac{1}{\varepsilon}\, I \cdot y \\ \frac{dy}{dt} &= -\frac{1}{\varepsilon^2}\, y + \frac{1}{\varepsilon}\, \xi(t) \end{aligned} \tag{4.33}$$

with

$$\langle \xi(t)\, \xi(0) \rangle = \delta(t)$$

$y(t)$ is exponentially correlated on a time scale ε^{-2}. The joint process is still a Markov process, with a Fokker–Planck equation for the joint probability

$W(I, y, t)$. Since we are not interested in any correlations between I and y explicitly, it would be entirely sufficient for all applications to calculate the reduced distribution $W(I, t)$, integrated over y:

$$W(I, t) = \int P(I, y, t)\, dy \tag{4.34}$$

At present, we are only interested in the limit of large but finite-bandwidth noise i.e., $\varepsilon \ll 1$. In this limit it is possible to eliminate the dummy variable y explicitly and derive a closed equation for $W(I, t)$ alone[29,30]:

$$\begin{aligned} \partial W(I, t)\, \partial t = & -\frac{\partial}{\partial I}(d - I)\, IW + \frac{\partial}{\partial I} I \frac{\partial}{\partial I} I(1 - \beta I)\, W \\ & + \varepsilon^2 D\left(I, \frac{\partial}{\partial I}\right) \delta(I - I^0)\, \delta(t) \end{aligned} \tag{4.35}$$

with

$$\beta^{-1} = d + \varepsilon^{-2}, \qquad D\left(I, \frac{\partial}{\partial I}\right) = \frac{\partial}{\partial I}\left(dI - I^2 - I\frac{\partial}{\partial I} I\right)$$

For simplifying the expressions we have rescaled the variables: $2d/Q \to d$ and $\varepsilon^2/Q \to \varepsilon^2$. This reduced equation differs in an essential way from a genuine Fokker–Planck equation since it does not establish a Markov process on the reduced phase space. Due to the elimination of the auxiliary variable y, the equation for W does not properly reproduce the short-time behavior either. For this reason, the initial conditions have to be renormalized properly, in order to account for the rapid transients in the time interval ε^{-1}. This is provided in Eq. (4.35) automatically by the inhomogeneity. The diffusion constant of this process is not positive and assumes negative values for $I > \beta^{-1}$. However, this regime is not accessible for a random trajectory that originates in the physical interval $0 < I < \beta^{-1}$, since at the boundary of this regime, the deterministic forces are repulsive and the noise vanishes.

The stationary solution of the evolution equation is easily obtained in the form

$$W_0(I) = \beta^d \frac{\Gamma(d + 1/\varepsilon^2)}{\Gamma(d)\, \Gamma(1/\varepsilon^2)}\, I^{d-1}(1 - \beta I)^{1/\varepsilon^2 - 1} \tag{4.36}$$

Also, the dynamic process is readily solved when we recognize that the eigenvalue problem corresponding to Eq. (4.35) can be transformed into the defining equation of the Gauss hypergeometric functions. By applying the appropriate integrability conditions, we find

$$W_m(I) = NI^{d-m-1}(1-\beta I)^{1/\varepsilon^2-1}\,{}_2F_1(-m, \beta^{-1}-m; d-2m+1; \beta I) \tag{4.37}$$

and the eigenvalues are

$$\lambda_m = m(d-m) \qquad \text{for} \quad 0 < m < m_0$$
$$\text{where} \quad m_0\colon \quad d-2 < 2m_0 < d$$

The normalization constant follows from

$$\int_0^{1/\beta} \frac{W_m^2(I)}{W_0(I)}\, dI = 1 \tag{4.38}$$

The stationary properties of this process are obtained quite easily and we find for the moments of the intensity in general

$$\langle I^n \rangle = \beta^{-n} \frac{\Gamma(d+n)\,\Gamma(d+1/\varepsilon^2)}{\Gamma(d)\,\Gamma(d+n+1/\varepsilon^2)} \tag{4.39}$$

and especially

$$\langle I \rangle = d$$
$$\langle I^2 \rangle = d(d+1)\frac{1+d\varepsilon^2}{1+\varepsilon^2+d\varepsilon^2}$$
$$\frac{\langle \Delta I^2 \rangle}{\langle I \rangle^2} = \frac{1}{d}(1+\varepsilon^2+d\varepsilon^2)^{-1}$$

These results obviously generalize the previous white noise expressions and in the limit $\varepsilon \to 0$ we recover the old results again. The photon counting distribution is found by direct integration to be of the form

$$W(n, T) = \frac{1}{n!}\left(\frac{\eta T}{\beta}\right)^n \frac{\Gamma(d+1/\beta)\,\Gamma(d+n)}{\Gamma(d)\,\Gamma(d+n+1/\varepsilon^2)}\,{}_1F_1\left(d+n; d+\frac{1}{\varepsilon^2}; -\frac{\eta T}{\varepsilon^2}\right) \tag{4.40}$$

Besides the discrete spectrum of eigenvalues there also exists a continuum, which close to threshold dominates the relaxation again. The correlation functions, however, are hardly manageable and one has to expand the exact expressions above. In leading order in ε one can calculate the relaxation of the transient moments and the correlation functions (Fig. 10). But even then, a final quadrature is left for numerical evaluation and we do not arrive at compact analytical expressions.

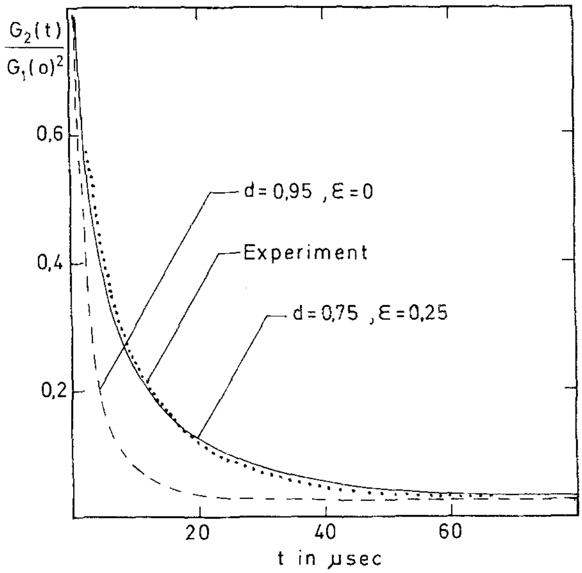


Fig. 10. Normalized intensity correlation function with nonwhite noise.

4.2. Subharmonic Generation and Quantum Noise

The previously discussed problems could all be formulated in terms of a classical stochastic process irrespective of the origin of noise. The Fokker–Planck equation of the traditional laser model was actually the equation for Glauber's P-representation describing spontaneous emission noise. The other models were purely classical from the beginning, since the noise was macroscopic and mostly imposed from the outside. In this section we want to discuss a model that exhibits the typical features of quantum noise, which cannot be formulated as a classical stochastic process. The interaction of light through a χ^2 nonlinearity in a crystal is known as the parametric oscillator. In case of degeneracy this process is called sub- or second-harmonic generation. This model is ideally suited to investigate the peculiar features of quantum noise and to demonstrate the usefulness of the mathematical methods presented above.

We will describe the parametric interaction of two quantized field modes b_1 and b_2 with the frequencies ω and 2ω, contained in a doubly resonant cavity with the corresponding loss rates γ_1 and γ_2. These modes are excited from the outside by two almost resonant driving forces F_1 and

F_2, where F_2 is the harmonic of F_1. The corresponding Hamiltonian can be written in the form

$$H = \sum_{n=1}^{2} \hbar\omega_n b_n^\dagger b_n + \frac{i\hbar}{2}\chi(b_1^{\dagger 2} b_2 - b_2^\dagger b_1^2)$$
$$\times\, i\hbar(b_1^\dagger F_1 e^{-i\omega t} - b_1 F_1^* e^{+i\omega t}) + i\hbar(b_2^\dagger F_2 e^{-2i\omega t} - b_2 F_2^* e^{+2i\omega t}) \qquad (4.41)$$

The time evolution of this dissipative dynamical system is described by the master equation for the statistical operator $\rho(t)$:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho] + \gamma_1[b_1, \rho b_1^\dagger] + \gamma_1[b_1\rho, b_1^\dagger] + \gamma_2[b_2, \rho b_2^\dagger] + \gamma_2[b_2\rho, b_2^\dagger] \qquad (4.42)$$

A traditional way to handle this equation is to use a c-number representation for $\rho(t)$ like, e.g., Glauber's P-representation:

$$\rho(t) = \int |\alpha_1, \alpha_2> P(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*, t) < \alpha_1, \alpha_2|\; d^2\alpha_1\, d^2\alpha_2 \qquad (4.43)$$

Thereby the master equation is transformed into a partial differential equation for the quasiprobability P. The linearized form of this process has already been studied at the beginning in connection with the different aspects of quantum noise. In linear approximation this process did not lead to a classical Fokker–Planck equation and we expect that this is also the case for the full nonlinear model. The parametric process is an ideal candidate for the application of the positive P-representation, since the corresponding evolution equation contains derivatives only up to second order. Therefore it can be transformed into a genuine Fokker–Planck equation without further approximations. The probability density corresponding to the positive P-representation is defined on an eight-dimensional phase space, i.e.,

$\alpha_1, \alpha_2, \beta_1, \beta_2$ and their complex conjugates

and the statistical operator $\rho(t)$ is given by

$$\rho(t) = \int |\alpha_1, \alpha_2\rangle \frac{P(\alpha_1, \alpha_2, \beta_1, \beta_2, t)}{\langle B_1^* \beta_2^* \alpha_1 \alpha_2\rangle} \langle \beta_1, \beta_2|\; d^2\alpha_1\, d^2\alpha_2\, d^2\beta_1\, d^2\beta_2 \qquad (4.44)$$

When inserting this form into the master equation, we find the following Fokker–Planck equation[31,32]:

$$\frac{\partial P}{\partial t}=\left\{\frac{\partial}{\partial \alpha_1}\left[(\gamma_1+i\delta_1)\,\alpha_1-\chi\beta_1\alpha_2-F_1\right]+\left[\alpha_1\to\beta_1,\,\delta_1\to-\delta_1\right]\right.$$
$$+\frac{\partial}{\partial \alpha_2}\left[(\gamma_2+i\delta_2)\,\alpha_2+\frac{1}{2}\chi\alpha_1^2-F_2\right]+\left[\alpha_2\to\beta_2,\,\delta_2\to-\delta_2\right]$$
$$\left.+\frac{\chi}{2}\frac{\partial^2}{\partial \alpha_1^2}\alpha_2+\frac{\chi}{2}\frac{\partial^2}{\partial \beta_1^2}\beta_2\right\}P(\alpha_1,\beta_1,\alpha_2,\beta_2,t) \tag{4.45}$$

For an analytic point of view, this partial differential equation in eight dimensions is a nightmare and there is little hope for an analytical solution. Statistically equivalent to Eq. (4.45), however, is a set of eight nonlinearly coupled Langevin equations with noise only in the equations for $\alpha_1(t)$ and $\beta_1(t)$. Their form is easily deduced from the Fokker–Planck equation by standard rules. Starting for the moment with the classical counterpart of this process, i.e., $\beta=\alpha^*$,

$$\begin{aligned}\frac{d}{dt}\alpha_1&=-(\gamma_1+\delta_1)\,\alpha_1+\chi\alpha_2\alpha_1^*+F_1\\ \frac{d}{dt}\alpha_2&=-(\gamma_2+\delta_2)\,\alpha_2-\frac{\chi}{2}\alpha_1^2+F_2\end{aligned} \tag{4.46}$$

we want to emphasize that already the classical problem in four dimensions has a rich variety of instabilities and bifurcations when the strength of the external fields F_1, F_2 is gradually increased. We differentiate between the cases of sub- or second harmonic by setting $F_1=0$ or $F_2=0$, respectively. For the initial condition we choose the vacuum state. In the subharmonic case, i.e., $F_1=0$, the field α_2 builds up from zero under the action of the coherent force F_2. Due to the parametric coupling of the subharmonic field, α_1 would remain in the vacuum state if it were not for the quantum fluctuations that enter the equation for α_1. In this way α_1 builds up entirely from noise. It fluctuates randomly about zero when operated below threshold $F_2=\gamma_1\gamma_2/\chi$ and becomes more and more coherent when the pump field intensity is increased.

The Langevin equations for the quantum process in eight dimensions are quite similar to Eq. (4.46). We only have to replace α_j^* by $\to\beta_j$ and include the appropriate equations for β_j. Some care has to be taken when inserting the corresponding noise sources ξ_1 and ξ_2 from the Fokker–Planck equation into the equations for $\alpha_1(t)$ and $\beta_1(t)$.

The problem is solved numerically by integrating the Langevin equations under the influence of randomly chosen forces $\xi_1(t)$ and $\xi_2(t)$. The sum over an appropriate number of trajectories approaches the quan-

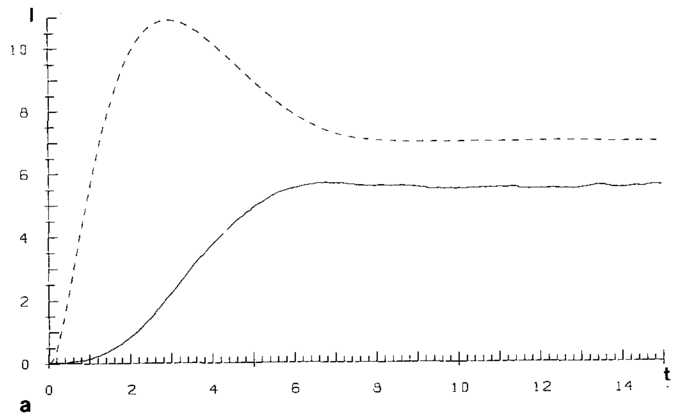


Fig. 11. Photon numbers for subharmonic generation with $F_1=0$, $F_2=0.75$. (—) the fundamental; (- -) the harmonic mode.

tum mechanical ensemble average. The temporal evolution of the intensities $\langle b_1^\dagger b_1\rangle=\langle\alpha_1\beta_1\rangle$ and $\langle b_2^\dagger b_2\rangle=\langle\alpha_2\beta_2\rangle$ rising up from the vacuum is shown in Fig. 11. The nonclassical feature of this process becomes evident when we calculate also the variances of the fields. Starting from the vacuum state with equal uncertainties, the variances of the two quadrature components become nonsymmetric in the course of time, indicating the creation of partially squeezed states. This is shown in Fig. 12. Since we are simulating a quantum mechanical process by purely classical stochastic trajectories, one might wonder if such an approach will always obey the

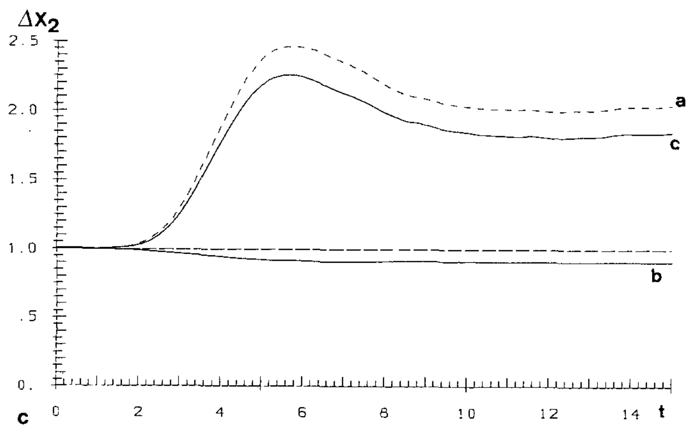


Fig. 12. Variances for the second harmonic field for the same parameters as in Fig. 11. (a) For the real part, (b) for the imaginary part. (c) The uncertainty product.

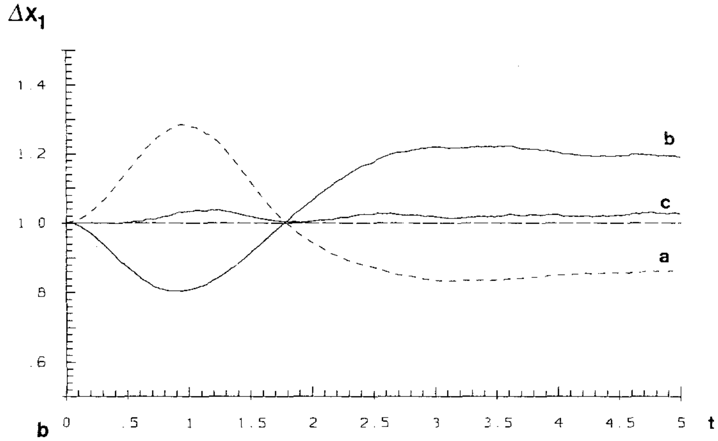


Fig. 13. Variances for the subharmonic field in the mixed case: $F_1 = 1$, $F_2 = 0.5$. The labeling of the curves is otherwise identical to Fig. 12.

restrictions of the uncertainty principle. This doubt is ruled out through this figure, where we have also plotted the uncertainty product. A more critical case in this respect is shown in Fig. 13, where a mixed state is discussed, i.e., both fields are driven from the outside simultaneously. Here squeezing and antisqueezing are exchanged in the course of time, and the trajectories cross "dangerously" close to the limit of the uncertainty product. Nevertheless, even in this case the uncertainty relation is satisfied for all times.

This example generalizes the linear model into the nonlinear regime, where hardly any analytic approach is known to solve such a problem. It also demonstrates the power and the convenience of the positive P-representation approach. It is obvious that this method can be applied also to other nonlinear quantum optical problems. However, at this point it should bot be concealed that there also exist some not yet understood problems associated with this method. When the corresponding classical process becomes unstable and undergoes a bifurcation to a limit cycle, to period doubling, or even to chaotic motion, more and more trajectories tend to exhibit large excursions from the average and may even diverge numerically. This is an unsolved problem at the moment and further work is needed to understand the significance of these excursions.

4.3. Photon Statistics of Quantum Jumps

In the early days of quantum mechanics it was possible to calculate the stationary properties of atoms such as the energy spectrum and the eigenfunctions. The irreversible transitions between those states—the quantum jumps—had to be postulated *ad hoc*, causing a lively controversy for years. Since the theory was based solely on Schrödinger's wave function and its unitary time evolution, the concept of jumps seemed odd and did not fit into the theoretical framework. When it was discovered that spontaneous emission was caused by the vacuum fluctuations of light, the physical origin of these jumps became understood and the concept of quantum jumps lost its artificial character. Also in all practical cases there is little if any trace of those individual jumps, since they are always averaged out in the presence of many atoms. In a different context, the idea of quantum jumps has come up again recently, because now it is possible to trap single atoms and to carry out those idealized experiments that had seemed to be mere *Gedanken* experiments a few years ago.[33,34]

When an electron is initially prepared in a long-living state, it will remain there for a certain period of time, until it eventually jumps back to the ground state by emitting a photon. This single event is rather difficult to see, since in a typical experiment only one of 10^3 photons is recorded. However, the trace of this individual event can be amplified by many orders of magnitude by coupling the ground state to a dipole-allowed excited state through a resonant laser field. Then no fluorescence from the allowed transition is observed as long as the electron is shelved in the metastable state. The return of the electron to the ground state manifests itself in a sudden onset of fluorescence. This is an easily observed signal, even from a single atom. The jump can be repeated in a stationary fashion, when a second resonant laser is used to drive the forbidden transition as well. Each downward jump from the metastable state is associated with the onset of fluorescence and any upward jump will quench the signal again. If this intuitive but oversimplified picture is basically correct, then it is possible to observe each individual jump. While the jump along the forbidden transition creates only a single spontaneous photon per lifetime, it is associated here with the random appearance and extinction of a strong fluorescence signal. Only the fluorescence from a single atom will show this effect. In the presence of a large number of atoms the jumps will be averaged out, since they occur randomly in time.

At first sight, the theoretical description of this effect seems to be rather difficult. It is a unique single-atom effect and we are interested in the properties of a single trajectory $I(t)$ only. Ensemble averages in the usual sense are not of great help, since they cannot predict jumps that occur randomly in time. Fluorescence is also a quantum mechanical effect and semiclassical theory is not applicable either.[35–38] The statistical properties of such a randomly fluctuating fluorescence signal can be characterized, however, by the photon counting probability $W(n, T)$. While $\langle n \rangle =$

$\sum nW(n, T)$ measures the average number of photons recorded in a time T, $W(n=0, T)$ is a measure for observing no event over T seconds. To be more specific, let us assume that the metastable state lives on the average for $\gamma_1^{-1} = 1$ sec and the allowed transition has a bandwidth of $\gamma_2 = 10^8$ sec^{-1}. In a collection time of $T \simeq 1$ sec we have on the average $\langle n \rangle \simeq 10^8$. If no jumps occur at all, then $W(n=0, T=1$ sec$)$ is of the order $\exp(-10^8)$, i.e., practically zero. The appearance of dark periods in the fluorescence signal due to the jumps requires that $W(n=0, T=1$ sec$) \simeq 1/3$, i.e., of order one, when both transitions are driven in saturation.[36] Dehmelt, who had suggested this idea almost 10 years ago, was also the first to observe the quantum jumps experimentally,[39] followed by other groups.[40–42]

In order to evaluate $W(n, T)$ in a consistent quantum mechanical form, in contrast to the semiclassical examples above, it is necessary to calculate the entire hierarchy of intensity correlation functions up to arbitrary order:

$$G_n = \langle b^\dagger(t_1)\, b^\dagger(t_2) \cdots b^\dagger(t_n)\, b(t_n) \cdots b(t_2)\, b(t_1) \rangle \tag{4.47}$$

The leading two-photon correlation function $G_2(t)$ already indicates the existence of dark periods in the fluorescence signal. γ_1 and γ_2, $\gamma_1 \gg \gamma_2$ are the spontaneous emission rates:

$$\frac{G_2(t)}{G_1(0)} = \frac{1}{3}\left\{1 + 1/2\left(\exp\left(-\frac{3}{2\gamma_2 t}\right) - 3\exp(-2\gamma_1 t)\right)\right\} \tag{4.48}$$

This result is derived under the assumption that the phase coherence is not of central importance and rate equations can be used in the limit of strong saturation. Two signatures of the quantum mechanical origin of this result are obvious: the curve rises from zero to 1/2 on the short time scale γ_1^{-1}, which is the well-known antibunching effect, and then falls off for larger times γ_2^{-1} from 1/2 to 1/3, indicating the presence of dark periods.

In the rate equation limit, it is also possible to calculate the intensity correlations of arbitrary order analytically and the counting distribution $W(0, T)$ is obtained, in closed form. For simplicity we assume that the allowed transition is saturated, while the rate R_2 of excitation into the metastable state can be varied. The probability of observing no photon over the entire time interval T is given by the following expression[36,37]:

$$W(0, T) = \frac{\gamma_2 + R_2}{2\gamma_2 + 3R_2}\left\{\frac{R_2}{\gamma_2 + R_2}\, e^{-(\gamma_2 + R_2)T} + 2e^{-\gamma_1 \eta T/2}\right\} + O\left(\frac{1}{R_1^2}\right) \tag{4.49}$$

The terms of order $1/R_1^2$ will have to be retained when it comes to calculate the entire dark-time statistics by differentiation. This curve is shown in

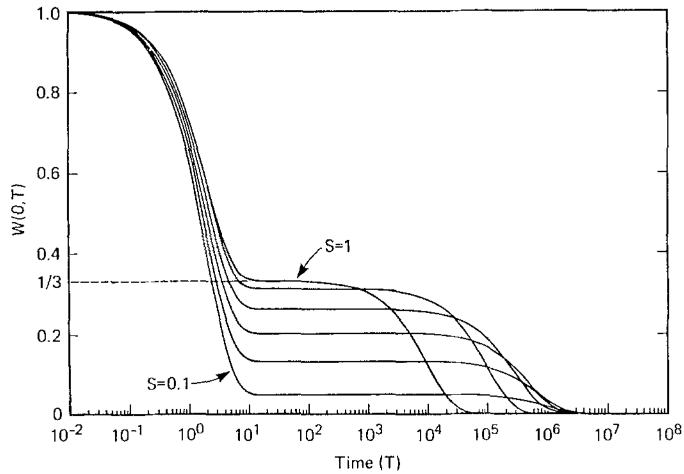


Fig. 14. Zero count probability $W(n=0, T)$ as a function of the interval T. The ratio of γ_1/γ_2 has been chosen to be 10^6. Here S is the saturation parameter.

Fig. 14. Quantum jumps must exist when even for a collection time of $T=1$ sec the probability $W(n=0, T=1 \text{ sec})$ is still of order unity, where on the average one expects to see 10^8 photons. In the interval

$$\gamma_1^{-1} \ll T \ll \gamma_2^{-1} \tag{4.50}$$

the counting probability yields

$$W(0, T) = \frac{R_2}{2\gamma_2 + 3R_2} \to \frac{1}{3} \tag{4.51}$$

which approaches a value of 1/3 in case of strong saturation, i.e., $R_2 \gg \gamma_2$ of the metastable level. This result leaves no doubts about the existence of quantum jumps in and out of the dark periods. It also describes their properties quantitatively. The probability of observing n events in a time T is obtained from the previous result by repeated differentiation with respect to η. For $n \neq 0$ we have

$$W(n, T) = \frac{2(\gamma_2 + R_2)}{2\gamma_2 + 3R_2} \frac{1}{n!} \left(\frac{\gamma_1 \eta T}{2}\right)^n \exp\left(-\frac{\gamma_1 \eta T}{2}\right) \tag{4.52}$$

This results in a Poisson distribution centered around the average photon count number, with the exception that the zero count probability is not included in this formula. $W(n, T)$ is a double-humped distribution with peaks at the two typical modes of operation, i.e., strong fluorescence and darkness. A further characterization of this process, which can be compared with experiments more easily, is the dark-time probability $P_2(T)\,dT$, defined above. By differentiation we find

$$P_2(T)=\frac{\gamma_2+R_2}{2\gamma_2+3R_2}\times\left\{R_2(\gamma_2+R_2)\,e^{-(\gamma_2+R_2)T}+\frac{(\gamma_1\eta)^2}{2}\,e^{-\gamma_1\eta T/2}-\left[R_2(\gamma_2+R_2)+\frac{(\gamma_1\eta)^2}{2}\right]e^{-2R_1T}\right\} \tag{4.53}$$

It distinguishes between the short dark times separating individual photons during fluorescence, which are statistically very numerous, and the rare but long dark periods when the electron is shelved. The initial rise of P_2 from zero is another indication of the antibunching effect typical for resonance fluorescence. Antibunching in the photon counting statistics and especially in $P(T)$ has not been discussed so far.

We can characterize the photon statistics also by its moments, the average photon number and its variance. In principle this requires the knowledge of $W(n, T)$ for arbitrary photon numbers. In general, it is quite difficult to derive a satisfactory expression already for $W(0, T)$ in analytic form. Such an explicit result, however, is needed to carry out the high-order derivatives ($\simeq 10^6$) that yield $W(n, T)$. Fortunately, the leading moments can be obtained directly from $W(0, T)$ and its low-order derivatives with respect to η:

The average photon number is given by[43]

$$\langle n\rangle=\sum_{n=0}^{\infty} nW(n, T)=-\eta\left(\frac{\partial}{\partial\eta}W(0, T)\right)_{\eta=0} \tag{4.54}$$

and the variance or the Mandel Q parameter is[43]

$$\frac{\langle\Delta n^2\rangle-\langle n\rangle^2}{\langle n\rangle^2}=Q=+\frac{\eta^2}{\langle n\rangle}\left\{\frac{\partial^2}{\partial\eta^2}W(0, T)-\left(\frac{\partial}{\partial\eta}W(0, T)\right)^2\right\}_{\eta=0} \tag{4.55}$$

Initial sub-Poissonian statistics is the natural result for most driven systems and we find quite generally

$$Q=-\eta T\rho_{11}(t=\infty)+\cdots \tag{4.56}$$

while for long times, Q assumes here very large values. This is in strong contrast to the two-level result, where $Q \to 0$ for $t \to \infty$. The sharp increase of Q for long times, i.e., $T \simeq \gamma_2^{-1}$,

$$Q \to \frac{2}{9} \eta \frac{\gamma_1}{\gamma_2} \qquad \text{for} \quad T \to \infty \tag{4.57}$$

is associated with enormous intensity fluctuations. This is also an unmistakeable trace of the intermittent fluorescence signal.

The question of coherence, which seems to be of basic interest here, has not been touched yet. In our intuitive understanding, as well as in our theoretical approach, we have disregarded the possibility of coherent superposition states and have focused only on the level populations. It is not intuitively obvious, however, how an atom will evolve with respect to spontanuous emission when it is initially prepared in a superposition of the two excited states. Will it be shelved or will it emit? To answer this question, it is necessary to derive the counting statistics based on the coherent quantum mechanical Bloch equations. In addition, one expects that rate equations characterize a single atom only poorly, since energy and phase relaxation occur on the same time scale. In the more general Bloch picture one can also hope to see some indication of the coherent Rabi oscillations in the counting statistics. It turns out that those oscillations are a quite subtle effect, which becomes visible only in the dark-time statistics $P(T)\,dT$.[43] Without presenting any details here, we only want to point out that quantum jumps or the intermittent fluorescence exist with or without coherence taken into account. While the quantitative details depend strongly on the relation of transverse to longitudinal relaxation rates, the mere existence of the jumps does not.

5. CONCLUSIONS

In quantum optical problems, noise can arise from various sources. It can result from a mere lack of control over the internal parameters of the system, can be imposed from the outside, or it can be an intrinsic property of the system, such as thermal or quantum noise.

Either form of noise can be seen in optical experiments, which makes it necessary to describe the light field and its temporal evolution in a statistical way. The typical differences of classical and quantum noise have been elucidated by some tutorial, linear examples, before we turned to the general question of the paper and presented a number of realistic and relevant nonlinear physical models where noise plays an essential role.

A common feature of most of these models was that to a large extent they could be solved in analytical form, which allowed us to compare the different models and their properties in a very compact and transparent way. The comparison of theoretical results with experimental observations should demonstrate that additive and multiplicative, classical and quantum noise are relevant for a proper understanding of quantum optical processes.

It was one of the aims of this paper to show that noise is not necessarily only an unavoidable nuisance limiting experimental accuracy; we hoped to demonstrate that fluctuations can also lead to rather interesting features, especially when quantum noise with its peculiar properties dominates over fluctuations of classical origin.

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